# Throttling numbers for cop vs gambler

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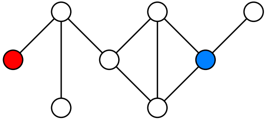
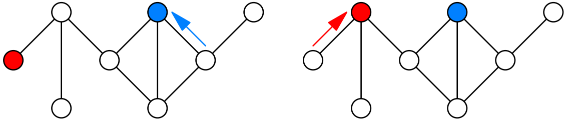
# Cop vs robber

- Game played on a simple, connected graph
- Two players: cop and robber.
- First, the cop picks a vertex, then the robber picks a vertex.

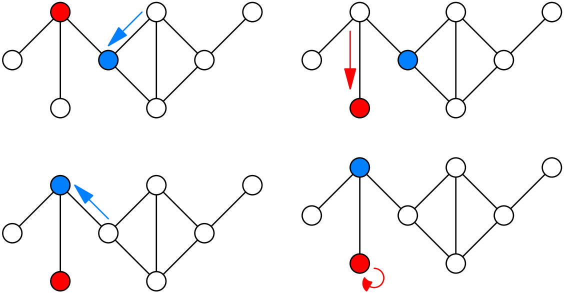


- They take turns, either moving to an adjacent vertex or staying put.

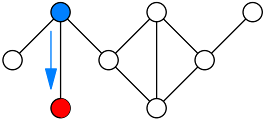


# Cop vs robber

- The cop wins if she can “capture” robber by moving to same vertex.

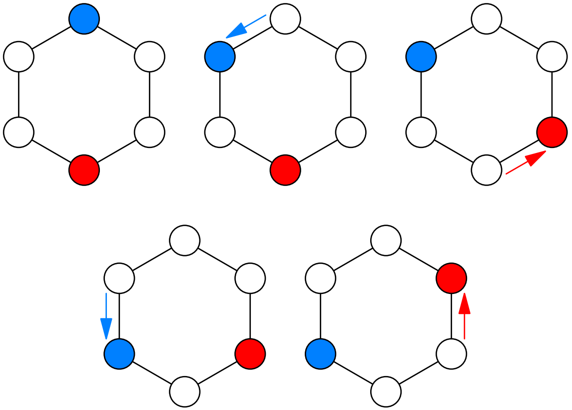


- Here, the cop wins next turn.



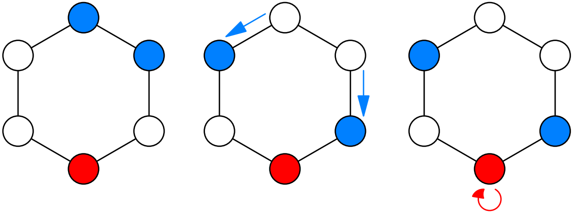
# More cops

- Sometimes one cop is enough to capture the robber, but not always.



# More cops

- But if there are *more* cops, they can always guarantee capture, no matter what the robber does.



- Since we can place cops on every vertex, there's a minimum number of cops such that they always win.

# Cop number

**Definition (Cop number)**

The *cop number* of a graph is the minimum number of cops needed to guarantee they win, no matter what the robber does.

- If one only cares about resources and not time, the cop number is a good way to measure the “cost” of capturing the robber.
- Later on we will see a way to incorporate both types of cost into our cost function.

**Example**

- Paths, complete graphs, and trees have a cop number of 1.
- Cycles of length at least four have a cop number of 2.

# Meyniel's conjecture

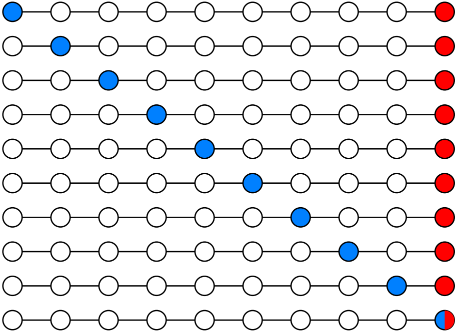
**Conjecture (Meyniel's conjecture)**

*The maximum cop number of a graph with n vertices is $O(\sqrt{n})$.*

- The $O(\sqrt{n})$ here means the cop number can be at most $2\sqrt{n}$, or $100\sqrt{n}$, or $k\sqrt{n}$ for some constant $k$.
- It's sharp: there are graphs of projective planes with cop number *at least* $\sqrt{\frac{n}{8}}$.
- It's notoriously hard: it's been conjectured since 1985, and we haven't even proved the cop number is $O(n^{1-\epsilon})$ for a fixed $\epsilon > 0$.

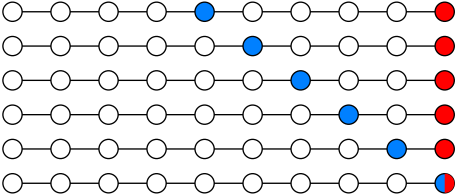
# Capture time

- Consider the minimum time it takes to guarantee capturing the robber.
- If we place a cop on the edge of a path with length $n$, it takes at most $n$ turns.



# Capture time

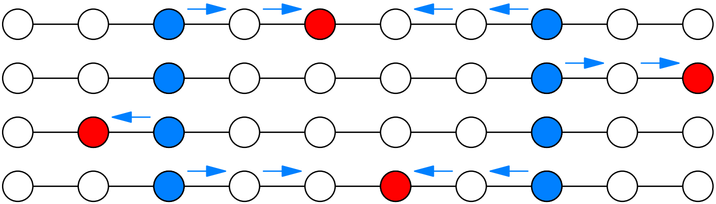
- But we can make it smaller if we place it in the middle instead.



- Here, we can guarantee capture in $\left\lceil \frac{n-1}{2} \right\rceil$ turns.
- So for one cop, the minimum guaranteed capture time is $\left\lceil \frac{n-1}{2} \right\rceil$.

# Capture time

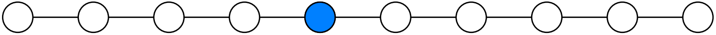
- If we had two cops and place them like this:



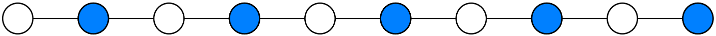
- No matter what the robber picks, it will take at most $\left\lceil \frac{n-2}{4} \right\rceil$ turns.
- It turns out this is optimal: for two cops, the minimum guaranteed capture time is $\left\lceil \frac{n-2}{4} \right\rceil$.

# Capture time

- There's a tradeoff: the more cops we have, the faster it takes to capture the robber.
- For the path, if we had one cop, the capture time is $\frac{n}{2}$.



- If we had $\left\lfloor \frac{n}{2} \right\rfloor$ cops, the capture time is 1.



- We want a kind of balance: a small number of cops, plus a quick capture time.

# Cop-throttling number

**Definition (Cop-throttling number)**

The *cop-throttling number* of a graph is the minimum of $k + \mathrm{capt}_k$, where $\mathrm{capt}_k$ is the minimum guaranteed capture time of $k$ cops.

- The $k$ and $\mathrm{capt}_k$ terms can be thought of as resources and time, respectively.
- Their sum gives a way to evaluate the "cost" of a strategy, where we have to balance time against resources.

**Example (Throttling number for the path)**

For a path with $n$ vertices, if you have $k$ cops, the minimum guaranteed capture time is $\left\lceil \frac{n-k}{2k} \right\rceil$. By the AM–GM inequality, for some $k$,

$$\sqrt{2n} - \frac{1}{2} < k + \left\lceil \frac{n-k}{2k} \right\rceil < \sqrt{2n} + \frac{1}{2}.$$

# Cop-throttling number

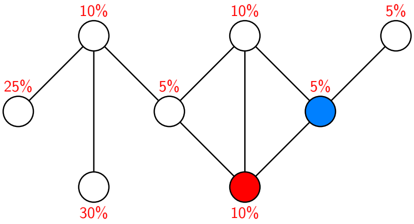
**Conjecture**

*The maximum cop-throttling number of a graph with $n$ vertices is $O(\sqrt{n})$.*

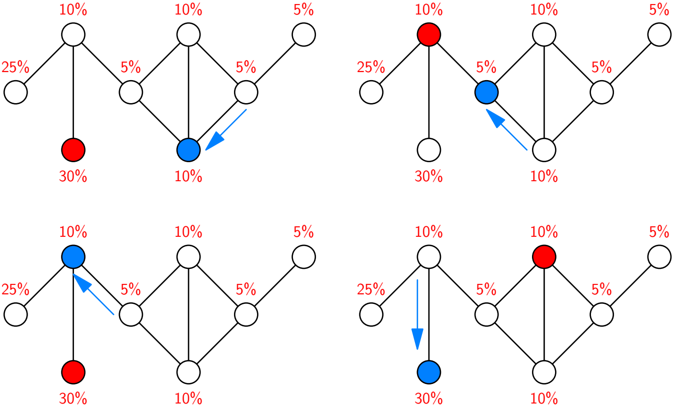
- We've just seen this is true for the path.
- If this was true, then Meyniel's conjecture has to be true as well, as the cop-throttling number is larger than the cop number.
- Unfortunately, this is false: some graphs have a cop-throttling number of at least $kn^{\frac{2}{3}}$ for some $k$.
- But it turns out this conjecture is true if we replace the robber with a "random" kind of robber. . .

# Cop vs gambler

- The cop and gambler game is played in rounds, not alternating turns.
- In the first round the cop picks any vertex and goes there.
- The gambler picks a *probability distribution* over the vertices, and goes to a vertex chosen by this distribution.

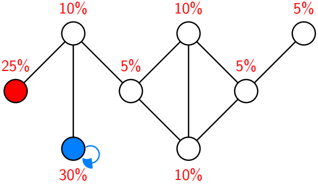


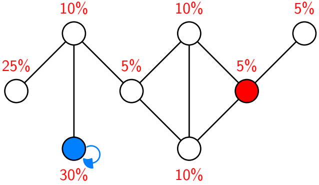
- In each subsequent round, the cop moves along an edge or stays put, and *simultaneously*, the gambler goes to a vertex chosen by his distribution.

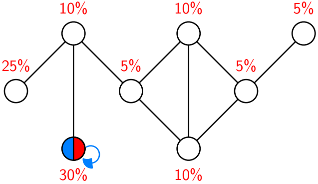


# Cop vs gambler

- Cop wants to minimize the expected (average) capture time, while gambler wants to maximize.



# Cop vs gambler

- In the *unknown* gambler, the cop doesn't know the distribution chosen by the gambler.
- In the *known* gambler, the cop does.
- In the known gambler, the cop can guarantee an expected capture time of at most $n$.
- The gambler can guarantee an expected capture time of at least $n$ (with the uniform distribution).
- In the unknown gambler, the cop can guarantee an expected capture time of at most $1.95335n$.

# Gambling Throttling Numbers

**Definition (Gambler throttling numbers)**

The *known gambler throttling number* of a graph is the minimum of $k + \text{capt}_k$, where $\text{capt}_k$ is the *expected* capture time of the known gambler with $k$ cops. The *unknown* gambler throttling number is defined similarly.

**Example**

- The known gambler throttling number of a graph with a Hamiltonian path is in the range $\left[2\sqrt{n}, \left\lceil 2\sqrt{n} \right\rceil\right]$.
  - The upper bound can be achieved when $\sqrt{n}$ cops patrol evenly distributed chunks along the path.
- For $n \gg 1$, the unknown gambler throttling number of a graph with a Hamiltonian cycle is in the range $\left[2\sqrt{n}, 2.0804\sqrt{n}\right)$.
  - The upper bound can be achieved when $\sqrt{n}$ cops remain evenly distributed around the cycle.
- The lower bounds can be achieved when the gambler uses the uniform distribution.

# Main Result

**Theorem**

*The (un)known gambler throttling number of a graph with $n \gg 1$ vertices is between $2\sqrt{n}$ and $3.96944\sqrt{n}$.*

- The known and unknown gambler throttling numbers grow with $n^{\frac{1}{2}}$ on all families of graphs.
- Compare with cop throttling number for the robber, which can grow with $\Omega\left(n^{\frac{2}{3}}\right)$.

# Future research

The current constants for the (un)known gambler throttling number in front of $\sqrt{n}$ are 2 and 3.96944. Further research may include:

- Tightening these constants for the known and/or unknown gambler on general graphs,
- Finding them for specific graphs,
- Studying throttling for other adversaries.

# Acknowledgments

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